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Wave formation on a liquid layer for de-icing airplane wings

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Wave formation on a thin liquid layer used for de-icing air-plane wings is investigated by studying the stability of air flow over a liquid-coated flat plate at zero angle of incidence. The ratio of the viscosity of the liquid to that of air is very high (over half a million), and the Reynolds number based on liquid depth and air viscosity is of the order of a few thousand in actual practice. Under these circumstances the analysis gives two formulas, in closed form, for the growth rate and phase velocity of the waves in terms of the wavenumber and other relevant parameters, including the Froude number F representing the gravity effect and a parameter S representing the surface-tension effect. In the calculation, the wavenumber is not restricted in any way.

The wavenumber of the waves that one expects to observe is that for which the growth rate is the maximum. The instability is one in which the viscosity difference between the two fluids (air and liquid) plays the dominant role, and is of the kind found by Yih (1967).

1. Introduction

In wintertime in northern countries, ice formation or snow accumulation on airplane wings while the airplanes are on the ground poses a threat to safety of flight. The practice of de-icing consists in spraying on the wings a layer of non-Newtonian liquid which has a very high viscosity at low shear rates but lower viscosity at higher shear rates, so that it can stay on the wings for a long time while the airplane is at rest, but is blown off after the airplane is in flight. But during a period after take-off and before the liquid is finally blown off, waves are formed on the liquid, which may affect the aerodynamic behaviour of the wings. The instability of the flow of the fluids (air and liquid) responsible for this formation is therefore a problem of practical interest, and is the subject of this study. For simplicity we shall consider the wing as a flat plate at zero angle of incidence.

The analysis in this paper will show that the instability is one in which the difference in viscosity of the two fluids plays the dominant role, because it induces a jump in the velocity gradient. Thus it is of the kind found by Yih (1967) and further investigated by Li (1969) and Hickox (1970), among other later workers, for long waves, and by Hooper & Boyd (1983), for waves not necessarily long. This study differs from the long-wave treatments in that no restriction is placed on the wavenumber, and that the Reynolds number is assumed large compared with unity. It differs from the work of Hooper & Boyd in that the problem chosen not only does not require infinite velocity at boundaries infinitely far away, but also is of some practical interest, and in that the viscosity ratio is very high, allowing results for the growth rate and the phase velocity of the disturbance to be obtained in closed formulas in terms of the wavenumber and the other relevant parameters, including those representing gravity and surface-tension effects.

Since the liquid is non-Newtonian, it was thought at first that the nonlinear constitution equations of the liquid should be constructed and used. The construction was done by using the table of viscosity variation with shear rate provided by the manufacturer of the de-icing liquid (Hoechst 1704), and by assuming the simplest tensorially consistent forms of the constitutive equations – actually by allowing the viscosity to contain invariants of the rate-of-deformation tensor. When the result was used in the equations of motion we found on elimination of the pressure terms a fourth-order differential equation like the Orr–Sommerfeld equation, but with additional terms. However, these terms turned out to be very small, as a result of the smallness of the rates of deformation in the primary flow of the liquid. We were therefore spared the pain of dealing with the non-Newtonian nature of the liquid, and could simply treat as constant the viscosity at the prevailing shear rate of the primary flow. This simplifies matters considerably.

2. The primary flow

We consider Blasius flow over a horizontal flat plate at zero angle of incidence, shown in figure 1. The free-stream velocity is denoted by $\hat{U}_0$, X is measured along the plate form its leading edge, and Y measured vertically upward from the undisturbed interface. We shall ignore the variation of the liquid depth d with X in the determination of the primary flow.

The viscosity of the air flow will be denoted by μ, and that of the liquid by μ_2. The ratio

$$m = \mu_2/\mu \tag{1}$$

is very large, and consequently (as will be shown later) the interfacial velocity $\hat{U}_s$ is very low. Then the flow of the air is just Blasius flow. The velocity of air in the direction of increasing X, measured in units of $\hat{U}_0$, is

$$U_1 = f'(\eta), \tag{2}$$

where f satisfies

$$2f''' + ff'' = 0,$$

and

$$\eta = (\hat{U}_0/\nu X)^{\frac{1}{2}} Y, \tag{3}$$

with $\nu = \mu/\rho$ denoting the kinematic viscosity of air and ρ its density.

Since, as is well known,

$$f''(0) = 0.332,$$

the shear stress at the interface is

$$\tau_0 = 0.332 \mu \hat{U}_0 (\hat{U}_0/\nu X)^{\frac{1}{2}}. \tag{4}$$

Since the depth d of the liquid is assumed constant, the velocity distribution in the liquid is linear, and if $\hat{U}_s$ denotes the velocity of the interface, continuity of shear stress demands

$$\mu_2 \hat{U}_s/d = \tau_0, \tag{5}$$

which determines $\hat{U}_s$. The dimensionless velocity gradient in the liquid is

$$a_2 = (\hat{U}_s/\hat{U}_0) \times 1 = \hat{U}_s/\hat{U}_0 = U_s, \tag{6}$$

which can be shown to be very small for practical cases of interest. In (6), U_s is the velocity at the interface in units of $\hat{U}_0$.

For the kind of instability we have in mind, the variation of U_1 near the interface

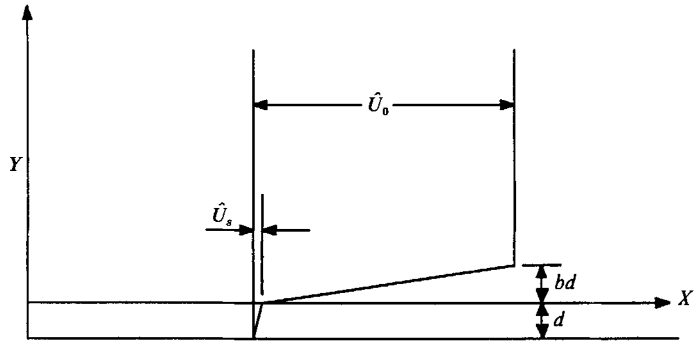

FIGURE 1. Definition sketch.

is the most important, and the curvature of U_1 given by (2) is zero at $Y = 0$, so that we can profitably replace (2) by

$$\left.\begin{aligned} U_1 &= Y/bd \quad \text{for} \quad 0 \leqslant Y \leqslant bd, \\ U_1 &= 1 \quad \text{for} \quad Y \geqslant bd. \end{aligned}\right\} \tag{7}$$

The boundary-layer thickness is thereby replaced by bd, where b is a dimensionless number defined by

$$\left.\frac{\partial U_1}{\partial Y}\right|_{Y=0} (bd) = 1, \tag{8}$$

in which (recall that U_1 is in units of U_0)

$$\frac{\partial U_1}{\partial Y} = 0.332\, R_X^{\frac{1}{2}}/X$$

by virtue of (4), with

$$R_X = \hat{U}_0 X/\nu. \tag{9}$$

As to the liquid, its velocity in units of U_0 is

$$U_2 = a_2 Y/d. \tag{10}$$

Introducing the dimensionless coordinates

$$x = X/d, \quad y = Y/d, \tag{11}$$

we can write (7) and (10) as

$$\left.\begin{aligned} U_1 &= a_1 y \quad \text{for} \quad 0 \leqslant y \leqslant b, \quad \text{with} \quad a_1 = b^{-1}, \\ U_1 &= 1 \quad \text{for} \quad y \geqslant b, \end{aligned}\right\} \tag{12}$$

and

$$U_2 = a_2 y. \tag{13}$$

3. Formulation of the stability problem

The flow is nearly parallel. For any value of x, we shall assume the flow to be a parallel one, with the velocity distribution given by (12) and (13), in which a_1 and a_2 are henceforth treated as constant, and not varying with x. This is the standard procedure for treating nearly parallel flows. In it, the origin of x need not be at the leading edge of the flat plate, but may be taken at the section where the stability of the flow is being studied.

The velocity components u and v (both in units of U_0) in the directions of increasing x and y satisfy the continuity equation

$$u_x + v_y = 0,$$

in which subscripts indicate partial differentiation. If the pressure p is measured in units of ρU_0^2 and the time t in units of d/U_0, the Navier–Stokes equations are, for air,

$$\frac{\mathrm{D}u}{\mathrm{D}t} = -p_x + R^{-1}\nabla^2 u, \tag{14}$$

$$\frac{\mathrm{D}v}{\mathrm{D}t} = -p_y - F_0^{-2} + R^{-1}\nabla^2 v, \tag{15}$$

in which

$$R = \hat{U}_0 d/\nu, \quad F_0 = \hat{U}_0/(gd)^{\frac{1}{2}} \tag{16}$$

are the Reynolds number and Froude number, respectively, and

$$\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}, \quad \nabla^2 = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial y^2}.$$

We shall, as usual, resolve the flow into its primary and perturbation parts. For air,

$$u = U_1 + u', \quad v = v', \quad p = P + p',$$

in which P is the dimensionless pressure for the primary flow, assumed independent of x. Then, since

$$u'_x + v'_y = 0,$$

we can introduce the stream function ψ, and write

$$u' = \psi_y, \quad v' = -\psi_x. \tag{17}$$

Assume

$$(\psi, p') = \{\phi(y), f(y)\} \exp \mathrm{i}\alpha(x - ct), \tag{18}$$

where

$$c = c_r + \mathrm{i}c_i \tag{19}$$

is the eigenvalue. For given R, F_0, and other relevant parameters, we seek to determine the α which makes the growth rate

$$\sigma_i = \alpha c_i \tag{20}$$

the maximum. With (17) and (18), (14) and (15) become, upon linearization,

$$\mathrm{i}\alpha\{(U_1 - c)\phi' - U_1'\phi\} = -\mathrm{i}\alpha f + R^{-1}(\phi''' - \alpha^2\phi'), \tag{21}$$

$$\alpha^2(c - U_1)\phi = f' + (\mathrm{i}\alpha/R)(\phi'' - \alpha^2\phi), \tag{22}$$

in which primes indicate differentiation with respect to y. Note that for the primary flow

$$-P_y - F_0^{-2} = 0,$$

which allows us to obtain (22) from (15). Since U_1' is the constant a_1, (21) and (22) give the Orr–Sommerfeld equation

$$\phi^{\mathrm{iv}} - 2\alpha^2\phi'' + \alpha^4\phi = \mathrm{i}\alpha R(U_1 - c)(\phi'' - \alpha^2\phi). \tag{23}$$

For the liquid, we choose to retain the meanings of R and f, and to write χ for ϕ. Then (21) and (22) become

$$\mathrm{i}\alpha r\{(U_2 - c)\chi' - U_2'\chi\} = -\mathrm{i}\alpha f + (m/R)(\chi''' - \alpha^2\chi'), \tag{24}$$

$$\alpha^2 r(c - U_2)\chi = f' + (\mathrm{i}\alpha m/R)(\chi'' - \alpha^2\chi), \tag{25}$$

and the equation corresponding to (23) is

$$\chi^{\mathrm{iv}} - 2\alpha^2\chi + \alpha^4\chi = \mathrm{i}\alpha R m^{-1} r (U_2 - c)(\chi'' - \alpha^2\chi), \tag{26}$$

in which

$$r = \rho_2/\rho \tag{27}$$

is the density ratio, with ρ_2 denoting the density of the liquid.

We now turn to the boundary conditions. For clarity let ϕ be denoted by ϕ_0 in the free stream ($b \leqslant y$), and by ϕ_1 for the boundary layer ($0 \leqslant y \leqslant b$). For the upper fluid, then,

$$\phi_0 \rightarrow 0 \quad \text{as} \quad y \rightarrow \infty. \tag{28}$$

At $y = b$, U_1 given by (12) is not analytic, and four conditions should be imposed there. For this purpose let η_1 be the displacement of the (artificial) lower boundary of the free stream, when disturbed, from $y = b$. Then the kinematic condition at $y = b$ is

$$(\eta_1)_t + (\eta_1)_x = -\mathrm{i}\alpha\phi(b)\exp \mathrm{i}\alpha(x - ct),$$

since $U_1(b) = 1$. Thus

$$\eta_1 = \frac{\phi(b)}{c-1}\exp \mathrm{i}\alpha(x - ct). \tag{29}$$

The continuity of velocity then demands

$$\phi_0(b) = \phi_1(b), \tag{30}$$

$$\phi_0'(b) - \phi_1'(b) = \frac{\phi_0(b)}{b(c-1)}, \tag{31}$$

since $U_1' = b^{-1}$ for the boundary layer. Continuity of shear stress at $y = b$ demands

$$\phi_0''(b) + \alpha^2\phi_0(b) = \phi_1''(b) + \alpha^2\phi_1(b), \tag{32}$$

and continuity of normal stress (see Yih 1967, equations (28)–(30), for derivation) demands

$$\{-\mathrm{i}\alpha R(c-1)\phi_0' - \phi_0''' + \alpha^2\phi_0'\} + 2\alpha^2\phi_0' \\ = \{-\mathrm{i}\alpha R[(c-1)\phi_1' + a_1\phi_1] - \phi_1''' + \alpha^2\phi_1'\} + 2\alpha^2\phi_1' \tag{33}$$

at $y = b$, since there is no density difference across $y = b$, and no surface tension there. For (32), recall that in fact the mean shear stress has no jump at $y = b$.

For the liquid layer the no-slip conditions at the solid boundary are

$$\chi(-1) = 0, \quad \chi'(-1) = 0. \tag{34}$$

At the interface of air and liquid, where $y = 0$, the conditions are as derived in Yih (1967, p. 341), and are, upon writing ϕ for ϕ_1 for brevity,

$$\phi(0) = \chi(0), \tag{35}$$

$$\phi'(0) - \chi'(0) = \frac{\phi(0)}{c'}(a_2 - a_1), \tag{36}$$

$$\phi''(0) + \alpha^2\phi(0) = m\{\chi''(0) + \alpha^2\chi(0)\}, \tag{37}$$

$$-\mathrm{i}\alpha R(c'\phi' + a_1\phi) - (\phi''' - \alpha^2\phi') + 2\alpha^2\phi' + \mathrm{i}r\alpha R(c'\chi' + a_2\chi) \\ + m(\chi''' - \alpha^2\chi') - 2\alpha^2 m\chi' = \mathrm{i}\alpha R(F^{-2} + \alpha^2 S)\phi/c', \tag{38}$$

in which

$$F^{-2} = (r-1)F_0^{-2}, \quad S = T/\rho\hat{U}_0^2 d, \quad c' = c - a_2. \tag{39}$$

Since, as will be shown, a_2 is very small, we may put a_2 to zero, and write c for c' in (35)–(38). In (38), all variables are for $y = 0$. The T in (39) is surface tension.

The differential system governing stability of the flow then consists of differential equations (23) and (26), and boundary conditions (28), and (30)–(38). It seems at first sight daunting, since R is large and α is unrestricted. It is surprising and fortunate that an explicit analytical solution is possible, principally because m is extremely large. This will be shown in what follows.

4. A simplification

We shall first simplify boundary conditions (31)–(33), while keeping (30) intact, and show that the four boundary conditions at $y = b$ can be reduced to two, in which only the inviscid solutions of (23) appear.

The solution of (23) satisfying (28) is, for the free stream,

$$\phi_0 = A_{01}\exp(-\alpha y) + A_{02}\exp\{-\beta(y-b)\}, \tag{40}$$

where

$$\beta^2 = \mathrm{i}\alpha R(1-c)+\alpha^2, \tag{41}$$

since $U_1 = 1$ for the free stream. We have taken that root of β which has a positive real part. The c in (41) is very small and therefore negligible, as will be verified *a posteriori*. The solution for (23) for ϕ_1 is

$$\phi_1 = A_{11}\exp(-\alpha y) + A_{12}\exp(\alpha y) + A_{13}\exp\{\beta(y-b)\}. \tag{42}$$

This is only approximate, for (23) has one variable coefficient when applied to the boundary layer. But (42) is sufficiently accurate, for $\exp\{\beta(y-b)\}$ decreases toward zero as y decreases from b only a small distance, since β is large (because R is large). Thus the variability of U_1 will not have an appreciable effect on the development here. Using (40) and (42) in (30)–(32), we readily find that if A_{01}, A_{11}, and A_{12} are of $O(1)$ in magnitude, A_{02} and A_{13} are both of order $O(R^{-\frac{1}{2}})$ and their difference of order $O(R^{-1})$. Furthermore, substituting (40) and (42) into (33), we find that the brace on the right-hand side does not contain A_{02}, and in the brace on the other side the term containing A_{13} is of $O(R^{-\frac{1}{2}})$. The ϕ' and ϕ_1' are all of $O(1)$. Thus (33) becomes, upon division by $\mathrm{i}\alpha R$ and ignoring terms of $O(R^{-\frac{1}{2}})$, at $y = b$,

$$(c-1)\,\phi_0' = (c-1)\,\phi_1' + a_1\,\phi_1, \tag{43}$$

in which all viscous solutions (i.e. those with coefficient A_{02} or A_{13}) are dropped. Correspondingly, in

$$\phi_0(b) = \phi_1(b), \tag{44}$$

which is (30), one now need only use the inviscid solutions, as in (43). Note that (43) incidentally agrees with (31). That is because there is no jump either in density or in velocity of the primary flow.

5. Construction of the eigenfunction ϕ

The simplification of the boundary conditions at $y = b$ to (43) and (44), which consist now entirely of inviscid solutions, allows us to construct ϕ by forming its inviscid part first, and then adding to that the viscous part to take care of the interfacial conditions at $y = 0$.

To obtain the inviscid part, let

$$\phi_0 = A_0\,\mathrm{e}^{-\alpha y}, \tag{45}$$

$$\phi_1 = A_1\,\mathrm{e}^{\alpha y} + A_2\,\mathrm{e}^{-\alpha y}. \tag{46}$$

These are just simplified versions of (40) and (42), with the viscous parts therein omitted. Application of (43) and (44) give two equations relating the A, the solution of which gives

$$A_1 = \frac{1}{2\alpha b(1-c)} \mathrm{e}^{-2\alpha b} A_0, \quad A_2 = \left(1 - \frac{1}{2\alpha b(1-c)}\right) A_0, \tag{47}$$

so that

$$\phi_1(0) = A_1 + A_2 = \frac{1}{1-c}(1-\lambda-c) A_0, \tag{48}$$

$$\phi_1'(0) = \alpha(A_1 - A_2) = \left\{\frac{1}{b(1-c)} - \alpha\left(\frac{1}{1-c} + \lambda\right)\right\} A_0, \tag{49}$$

in which

$$\lambda = \frac{1}{2\alpha b}(1 - \mathrm{e}^{-2\alpha b}).$$

Note that

$$0 \leqslant \lambda \leqslant 1.$$

Assuming c to the very small compared with 1, we can write (48) and (49) as

$$\phi_1(0) = (1-\lambda) A_0, \tag{50}$$

$$\phi_1'(0) = \{b^{-1} - \alpha(1+\lambda)\} A_0. \tag{51}$$

These will be used in place of (48) and (49).

Equation (45) already satisfies condition (28), and (45) and (46) satisfy (43) and (44) if A_1 and A_2 are given by (47). We shall now add to ϕ_1 the viscous solution $A_3 \phi_3$ to from the eigenfunction ϕ for the boundary layer ($0 \leqslant y \leqslant b$). Thus,

$$\phi = \phi_1 + A_3 \phi_3, \tag{52}$$

in which (see Lin 1955, p. 40)

$$\phi_3 = \int_\infty^\eta \mathrm{d}\eta \int_\infty^\eta H_{\frac{1}{3}}^{(1)} \{\tfrac{2}{3}(\mathrm{i}\eta)^{\frac{3}{2}}\} \eta^{\frac{1}{2}} \,\mathrm{d}\eta, \tag{53}$$

where

$$\eta = \frac{y - y_c}{\epsilon}, \quad \epsilon = (\alpha R U_1')^{-\frac{1}{3}} = \left(\frac{b}{\alpha R}\right)^{\frac{1}{3}}, \tag{54}$$

where y_c is the value of y at which $U_1 = c$. Since c is expected to be very small, we can henceforth take $y_c = 0$. The subscript 3 in ϕ_3 is used to honour tradition. There is no ϕ_2. In (52), A_3 is not yet related to A_0. That relation awaits the application of the interfacial conditions.

We note in passing that neither ϕ_1 nor ϕ_3 is singular, nor indeed is ϕ_0. Since $U_1'' = 0$, the well-known Rayleigh equation loses its singular term, and the solutions ϕ_0 and ϕ_1, consisting of exponential functions, are exact. This is the great advantage of adopting (12) to replace (2). That ϕ_3 is not singular is of course well known. In computing ϕ_3, Holstein (1950) took advantage of the fact that R is large, and thereby ignored terms of smaller orders of magnitude to obtain the asymptotic form of the Orr–Sommerfeld equation, which he proceeded to solve to obtain ϕ_3. But that asymptotic form is not singular. Hence ϕ_3 is not singular.

6. Construction of the eigenfunction χ

We shall show that in cases of practical interest a_2 in (13) and therefore U_2 in (26) are very small compared with 1 since m is large. Furthermore it will be assumed (since we do not know c yet) that $\alpha R r m^{-1} c$ in (26) is very small since m is very large.

This assumption will be verified *a posteriori*. Under this assumption, then, the right-hand side can be neglected and the solution of (26) is

$$\chi = \mathrm{A}\cosh \alpha y + \mathrm{B}\sinh \alpha y + Cy\cosh \alpha y + Dy\sinh \alpha y. \tag{55}$$

Since m is very large, we shall simplify (37) to

$$\chi''(0) + \alpha^2\chi(0) = 0, \tag{56}$$

which gives

$$-\alpha A = D. \tag{57}$$

Applying (34), and eliminating B, we have

$$\alpha A = (\alpha - \sinh\alpha\cosh\alpha)\, C + (\sinh^2\alpha)\, D, \tag{58}$$

which combines with (57) to give

$$(\alpha\cosh^2\alpha)\, A = (\alpha - \sinh\alpha\cosh\alpha)\, C. \tag{59}$$

Finally $\chi(-1) = 0$ gives

$$(\cosh^2\alpha + \alpha^2)\, A = (\sinh\alpha\cosh\alpha - \alpha)\, \mathrm{B}. \tag{60}$$

Thus all the coefficients B, C, and D have been expressed in terms of A.

The following equalities will be useful later:

$$\chi(0) = A, \tag{61}$$

$$\chi'(0) = \alpha B + C = \frac{\alpha^3}{\sinh\alpha\cosh\alpha - \alpha} A, \tag{62}$$

$$\chi'''(0) - 3\alpha^2\chi'(0) = -\frac{2\alpha^3(\cosh^2\alpha + \alpha^2)}{\sinh\alpha\cosh\alpha - \alpha}\mathrm{A}. \tag{63}$$

We are now in a position to apply the interfacial conditions.

7. Calculation of the growth rate

Upon neglecting a_2 in the parentheses on the right-hand side of (36), elimination of $\phi(0)$ between (35) and (36) gives

$$-bc'\phi'(0)f = \chi(0) - bc'\chi'(0). \tag{64}$$

Let

$$\phi_3(0) = \beta, \tag{65}$$

$$(\mathrm{d}\phi_3/\mathrm{d}\eta)_{\eta=0} = \gamma, \tag{66}$$

where the β is not the same as that in (41), which is no longer needed. Holstein (1950, p. 36) gave

$$\beta = -0.8660 + 0.2320\mathrm{i}, \quad \gamma = 1.1154 + 0.2989\mathrm{i}.$$

Equation (64) then takes the form

$$-bc'[\{b^{-1} - \alpha(1+\lambda)\}A_0 + \epsilon^{-1}\gamma A_3] = A - \frac{bc'\alpha^3}{\sinh\alpha\cosh\alpha - \alpha}A. \tag{67}$$

On the assumption that c' is very small for large m (to be verified *a posteriori*), a glance at (60) reassures us that the last term in (67) is small compared with A, so that (67) can be written as

$$-c'[\{1 - \alpha b(1+\lambda)\}A_0 + b\epsilon^{-1}\gamma A_3] = A. \tag{68}$$

Equations (50), (52), and (61) enable us to write (35) as

$$(1-\lambda)A_0 + \beta A_3 = A. \tag{69}$$

Elimination of A_3 between (68) and (69) gives

$$c'\{-1+\alpha b(1+\lambda)+b\epsilon^{-1}\gamma\beta^{-1}(1-\lambda)\}A_0 = (1+b\epsilon^{-1}\beta^{-1}c')A.$$

Since $\epsilon^{-1}c'$ is still very small, and

$$\gamma\beta^{-1} = -(1.1155+0.6440\mathrm{i}),$$

this can be written as

$$-bc'(P_r+\mathrm{i}P_i)A_0 = A, \tag{70}$$

where (P is now not the pressure in the primary flow)

$$\begin{aligned} P &= P_r+\mathrm{i}P_i \\ &= b^{-1}-\alpha(1+\lambda)+-\gamma\beta^{-1}\epsilon^{-1}(1-\lambda). \end{aligned} \tag{71}$$

Equation (70) shows that the A in (69) can be neglected, since c' is small. Thus

$$(1-\lambda)A_0+\beta A_3 = 0,$$

or

$$\begin{aligned} A_3 &= -\beta^{-1}(1-\lambda)A_0 \\ &= \frac{1}{bc'(P_r+\mathrm{i}P_i)}\beta^{-1}(1-\lambda)A. \end{aligned} \tag{72}$$

We now turn to (38) to evaluate c' and $\sigma_i = \alpha c_i$.

Because a_2 is very small, (36) enables us to write (38) as

$$\mathrm{i}(r-1)\alpha Rc'\chi'+m(\chi'''-3\alpha^2\chi')-(\phi'''-3\alpha^2\phi')-\mathrm{i}\alpha R(F^{-2}+\alpha^2S)\chi/c' = 0, \tag{73}$$

all functions being evaluated at $y = 0$. All the terms involving χ have been given by (61), (62), and (63). It remains to evaluate the term containing ϕ, which is

$$\begin{aligned} \phi'''-3\alpha^2\phi' &= -2\alpha^3\phi_1'+A_3\{\epsilon^{-3}(\mathrm{d}^3\phi_3/\mathrm{d}\eta^3)-3\alpha^3\epsilon^{-1}\,\mathrm{d}\phi_3/\mathrm{d}\eta\} \\ &= -2\alpha^2\{b^{-1}-\alpha(1+\lambda)\}A_0+\epsilon^{-3}(-\mathrm{i}\beta-3\alpha^2\epsilon^2\gamma)A_3, \end{aligned}$$

where $\mathrm{d}^3\phi_3/\mathrm{d}\eta^3$ at $\eta = 0$ is found by integration by parts of the differential equation for ϕ_3, with ϕ_3 (0) given by (65).

Using (70) and (72), we have

$$\phi'''-3\alpha^2\phi' = -\frac{\alpha RA}{c'b^2}\frac{Q_r+\mathrm{i}Q_i}{P_r+\mathrm{i}P_i}, \tag{74}$$

where

$$Q = Q_r+\mathrm{i}Q_i = (1-\lambda)(\mathrm{i}+3\gamma\beta^{-1}\alpha^2\epsilon^2)-\frac{2\alpha}{R}\{1-\alpha b(1+\lambda)\}. \tag{75}$$

In the above calculation, recall that $\epsilon^3 = b/\alpha R$.

Examination of (61), (62), (63), and (74) shows that (73) is a quadratic equation in c'. But even the solution of a quadratic equation is not necessary. Because m is very large, the two roots of the quadratic equation are obtained for the first root by ignoring the last term, and for the second root by ignoring the first term. In the first case a large negative c_i is obtained, corresponding to strong damping. Any instability would have to come from the second case, which gives

$$\frac{2\alpha^2(\cosh^2 a+\alpha^2)}{\sinh\alpha\cosh\alpha-\alpha}\frac{b^2mc'}{R} = \left\{\frac{Q_r+\mathrm{i}Q_i}{P_r+\mathrm{i}P_i}-\mathrm{i}b^2(F^{-2}+\alpha^2S)\right\}, \tag{76}$$

or, with $c_r' = c_r - a_2$, $\sigma_i = \alpha c_i$,

$$\frac{b^2 m}{R} c_r' = \frac{P_r Q_r + P_i Q_i}{|P|^2} \frac{\sinh\alpha\cosh\alpha - \alpha}{2\alpha^2(\cosh^2\alpha + \alpha^2)}, \tag{77}$$

$$\frac{b^2 m}{R}\sigma_i = \left[\frac{P_r Q_i - P_i Q_r}{|P|^2} - b^2(F^{-2} + \alpha^2 S)\right]\frac{\sinh\alpha\cosh\alpha - \alpha}{2\alpha(\cosh^2\alpha + \alpha^2)}. \tag{78}$$

Equation (77) gives the phase velocity and (78) gives the growth rate. They are the main results of this study.

From (76), it is clear that the instability under study arises from the term Q/P. Examination of (71)shows that the dominant term in P is

$$-\gamma\beta^{-1}\epsilon^{-1}(1-\lambda),$$

since ϵ is small, and examination of (75) shows that the dominant term in Q is

$$i(1-\lambda).$$

Thus the dominant term in Q/P is $\quad i\beta\gamma^{-1}\epsilon$.

Since $$\beta = \phi_3(0), \quad \gamma = \phi_3'(0),$$

and ϕ_3 is the solution of $$\phi_3^{iv} = i\alpha R U_1'(0)(y-y_c)\phi_3'',$$

in which only the velocity gradient $U'(0)$ appears (and no other representatives of U_1), it is clear that the instability under study arises only from the velocity gradients of the fluids at their interface, and that the exact form of the velocity profile of the air and, in particular, its curvature (to which fluid dynamicists have historically attached an enormous importance), have little consequence here. The difference in the velocity gradients of the two fluids come from the difference in their viscosity, and therefore the instability under examination here is of the kind found by Yih (1967).

Note also that, since the m under consideration is roughly 250 times R, (76) presents no difficulty in determining c' when both m and R are large, because we do not regard them as tending toward infinity, but, rather, simply use their actual values.

In the next section we shall take a case for which some experimental data are available, and calculate the α for which σ_i is the maximum, for given values of the relevant parameters b, R, F, and S. This will be compared with the observed α. To make sure that this calculation will not be futile, however, it is timely to look at (78) more closely here. Examination of P defined by (71) and Q defined by (75) reveals that, since $\lambda \to 1$ as $\alpha \to 0$,

$$P \to b^{-1} \quad \text{and} \quad Q \to 0 \quad \text{as} \quad \alpha \to 0.$$

Thus the bracket in (78) becomes negative for small α, and the flow is stable. Furthermore, for fixed R and b

$$P \sim \alpha \quad \text{and} \quad Q \sim 2bR^{-1}\alpha^2 \quad \text{for large}\,\alpha,$$

so that again the bracket becomes negative. For small enough F^{-2} and S and intermediate α, instability is possible, and whenever there is instability a maximum σ_i is guaranteed. Note from the definitions of F^{-2} and S that they are small if $\hat{U}_0$ is large. Thus instability is likely at high velocities.

8. Application of the theory to a special case

Some experimental data on wave formation in a de-icing fluid on an experimental wing are available in a report by Hendrickson & Hill (1987, henceforth referred to as HH). The experiments were not elaborate, but the data give indications of the wavelengths of the waves, and the order of magnitude of their phase velocity. It seems desirable that (78) be applied to one of the experiments.

In these experiments, the de-icing fluid was the Hoechst 1704 liquid. In the experiment chosen for comparison, it was not diluted, and its viscosity (μ_2) near 0 °C or below and at the prevailing (very small) shear rate to which it was subjected, was near 10 Pa s. We shall use this figure for its viscosity. Since μ for air at -10 °C is 1.67×10^{-5} Pa s, the viscosity ratio m is 598802, which is very large indeed. Its surface tension at -10 °C is 31.3 mN/m, which will be used for T. The liquid–air density ratio r is 972. The free-stream speed was 53 knots, or 27.28 m/s. The chord length was 0.279 m.

We shall take for investigation the flow at the $\frac{3}{4}$-chord section, where $X = 0.2092$ m, because the uncertainty of depth seems less there and the waves seem more developed. The initial average depth of the liquid was 0.91 mm, but the data show considerable variation even in the initial depth. After 5.6 s, the depth differed from its initial value everywhere, and near the $\frac{3}{4}$-chord point figure 19 of HH gives (roughly) a mean depth of 1.1 mm. This will be the value taken for d.

The kinematic viscosity ν for air is 1.24×10^{-5} m^2/s. At $\frac{3}{4}$-chord, the Reynolds number R_X based on $X = 0.2092$ m and the free-stream speed is 460248. The shear stress at the interface, upon neglect of the interfacial velocity which will be shown to be small, is $\tau_0 = 29\,364.09\mu$. Thus, if $\hat{U}_1$ denotes the dimensional velocity in air,

$$\mathrm{d}\hat{U}_1/\mathrm{d}Y = 29\,364.09/\mathrm{s} \quad \text{at} \quad Y = 0,$$

or, in dimensionless terms, $\quad U_1' = 1.1841 \quad \text{at} \quad y = 0.$

Since $U_1' = b^{-1}$, we have $\qquad b = 0.8446.$

The Reynolds number R based on d, ν, and the free-stream velocity is

$$R = 2419.$$

For the liquid, $\qquad \mu_2 \mathrm{d}\hat{U}_2/\mathrm{d}Y = 29\,364.09\mu/\mathrm{s},$

or $\qquad \mathrm{d}\hat{U}_2/\mathrm{d}Y = 0.0490/\mathrm{s},$

This gives $\qquad a_2 = U_2' = \mathrm{d}U_2/\mathrm{d}y = 1.976 \times 10^{-6},$

which is very small indeed, as assumed. The dimensional interfacial velocity corresponding to this value of a_2 is

$$\hat{U}_s = a_2 \hat{U}_0 = 0.0539\,\mathrm{mm/s},$$

which is negligible, as assumed, for the purpose of calculating τ_0 (shear stress at the interface, for the primary flow). The values of F^{-2} and S can be readily computed.

Summarizing, we have, for free-stream speed 27.28 m/s, $d = 1.1$ mm, and $r = 972$,

$$R = 2419, \quad b = 0.8446, \quad m = 598\,802,$$
$$F^{-2} = 0.014101, \quad S = 0.27621.$$

With these parameters given, a brief calculation shows that the flow is stable for

$$\alpha \leqslant 0.01 \quad \text{or} \quad \alpha \geqslant 0.5 \qquad (79)$$

and is neutrally stable, or very nearly so, when the equality signs hold. The maximum σ_i occurs at

$$\alpha = 0.33, \tag{80}$$

at which (with $\sigma_i = \alpha c_i$)

$$\frac{b^2 m}{R}\sigma_i = 8.86 \times 10^{-4}, \quad \frac{b^2 m}{R} c_r' = 2.58 \times 10^{-3}. \tag{81}$$

Thus the assumption that c' $(= c_r' + \mathrm{i}c_i)$ is small is amply verified. The value of α of the waves observed by HH (pp. 29–30) is, by the best estimate from their figure 19 on p. 30,

$$\alpha = 0.5. \tag{82}$$

Thus the theoretical prediction of α is 34 % too low, but seems to be of the right order of magnitude.

Although HH did not measure c_r, their photographs taken at various times suggest that it is small, certainly far less than the c_r predicted for the classical Tollmien–Schlichting waves for Blasius flow, which is of the order of 0.2 (Shen 1954, see Schlichting 1960, p. 396), or at least 0.05 according to the earlier prediction of Tollmien (1929, see Schlichting 1960, p. 397). The α^* for the most unstable mode in the Tollmien– Schlichting theory for Blasius flow is approximately 0.26 (Shen 1954, or Schlichting 1960, p. 396), where α^* is the wavenumber based on the momentum thickness δ^*, which is 0.534 mm for the case at hand. This α^* corresponds to an α (based on d) of only 0.126, far less than the experimental value of 0.5. (Note that $\alpha^* = 0.26$ corresponds to a Reynolds number based on δ^* of 1174.) Thus the waves photographed by HH do not seem to be classical Tollmien–Schlichting waves.

9. Conclusion

With the assumptions that a_2 and c' are very small verified *a posteriori*, (77) gives the phase velocity and (78) the growth rate. The reasonable agreement between the theoretical and experimental values of the wavenumber of the most unstable mode suggests that the wave formation arises from the instability treated here, which is of the kind found earlier by Yih (1967), and is a result principally of the viscosity differences between air and the de-icing liquid.

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